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The Kantowski-Sachs Space-Time in Loop Quantum Gravity

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We extend the ideas introduced in the previous work to a more general space-time. In particular we consider the Kantowski-Sachs space time with space section with topology $R \times S^2$. In this way we want to study a general space time that we think to be the space time inside the horizon of a black hole. In this case the phase space is four dimensional and we simply apply the quantization procedure suggested by *loop quantum gravity* and based on an alternative to the Schroedinger representation introduced by H. Halvorson. Through this quantization procedure we show that the inverse of the volume density and the Schwarzschild curvature invariant are upper bounded and so the space time is singularity free. Also in this case we can extend dynamically the space time beyond the classical singularity.

KEY WORDS: quantum gravity; loop quantum gravity; quantum aspects of black hole; black hole singularity.

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1. INTRODUCTION

This work is a generalization of the recent results obtained for the Schwarzschild solution inside the horizon and near the singularity where the operator $1/r$ and so the curvature invariant $\mathcal{R}_{\mu\nu\rho\sigma}\,\mathcal{R}^{\mu\nu\rho\sigma} = 48M^2G_N^2/r^6$ are non divergent in the quantum theory. This work is suggested from a paper on Loop Quantum Cosmology (Ashtekar *et al.*, 2003; Bojowald, 2001a,b). In this paper we use the same non Schrödinger procedure of quantization used in the previous paper (Leonardo, 2004) and in the work of V. Husain and O. Winkler on quantum cosmology but introduced by Halvorson (2001) and also by Ashtekar *et al.* (2003).

In this paper we focus on a general two dimensional minisuperspace with space section of topology $\mathbf{R} \times \mathbf{S}^2$ which is know as Kantowski-Sachs space time (Christodoulakis, 2002; Halliwell and Louko, 1990; Kantowski and Sachs, 1966; Luca and Torrence, 1990). The Schwarzschild space time inside the horizon is a particular representative of this class of metrics. Using this method (Gambini

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et al., 1998; Thiemann, 1996, 1998a,b) we can define the inverse volume density and the Schwarzschild curvature invariant in term of the holonomy analog and the volume itself and we show that these quantity are finite and upper bounded. Using also the result in (Gambini *et al.*, 1998; Thiemann, 1996, 1998a,b) we can obtain the Hamiltonian constraint in terms of the volume and at the quantum level we have a discrete equation depending on two parameters for the coefficients of the physical states.

The paper is organized as follows: in the first section we report the metric we want to study; we consider this space time as the interior of a black hole in a more general form respect to the work (Leonardo, 2004). We calculate the hamiltonian constraint, the volume operator and we introduce the fundamental variables of the theory. In the second section we quantize the system using the non Schrödinger procedure of quantization (Ashtekar *et al.*, 2003; Halvorson, 2001; Viqar and Oliver, 2003). In this section we show that the inverse volume operator and a quantity inspired from the Schwarzschild curvature invariant are singularity free in quantum gravity. We conclude calculating the states that resolve the Hamiltonian constraint.

2. THE SPHERICALLY SYMMETRIC SPACE-TIME

We want to study a generical metric for an homogeneous, anisotropic space with spatial section of topology $\mathbf{R} \times \mathbf{S}^2$, this is the Kantowski-Sachs Space-Time. In this case we have two independent functions of the time $a(t)$ and $b(t)$ and the metric assume the following form

$$ds^2 = -dt^2 + a^2(t)dr^2 + b^2(t)(\sin^2\theta d\phi^2 + d\theta^2). \tag{1}$$

In the previous paper (Leonardo, 2004) we considered the Schwarzschild solution, so $a(t)$ was not a general function of t but it was a function of $b(t)$ which was the only independent function.

The Diff-constraint for the class of metrics in (1) is identically satisfied and the Hamiltonian constraint in terms of $\dot{a}$ and $\dot{b}$ is

$$H_L = |a|\,\dot{b}^2 + 2\,\dot{a}\,\dot{b}\,b\,\mathrm{sgn}(a) + |a|, \tag{2}$$

in terms of p_a and p_b is

$$H_c = \frac{G_N\,|a|\,p_a^2}{2R\,b^2} - \frac{G_N\,p_a\,p_b\,\mathrm{sgn}(a)}{Rb} - \frac{R}{2G_N}|a|. \tag{3}$$

The volume of a space section is

$$V = \int dr\,d\phi\,d\theta\,h^{1/2} = 4\pi R|a|b^2. \tag{4}$$

where R is a cutt—off on the space radial coordinate. We can work also with radial densities because the model is homogeneous and all the following results remain identical.[2] In another way, the spatial homogeneity enable us to fix a linear radial cell $\mathcal{L}_r$ and restrict all integrations to this cell (Ashtekar *et al.*, 2003; Bojowald, 2001a,b). We will consider this second possibility in the rest of this paper and we will take $R = l_P$. We have two canonical pairs, one is given by $a \equiv x_a$ and p_a, the other is given by $b \equiv x_b$ and p_b for which the Poisson brackets are $\{x_a, p_a\} = 1$ and $\{x_b, p_b\} = 1$. From now on we consider $x_a, x_b \in \mathbb{R}$ and we will introduce the modulus of x_a and x_b where it is necessary. This choice to take $x_a, x_b \in \mathbb{R}$ is not correct classically because we have a singularity in $b = 0$, but the situation can be (a priori) different in quantum theory; and it will be, as we will see.

Following (Leonardo, 2004) we introduce an algebra of classical observable and we write the quantities of physical interest in terms of these variable. As in loop quantum gravity (Ashtekar, 2004; Rovelli, 2004; Thiemann, 2001, 2003) we use the fundamental variables x_a, x_b and

$$U_{\gamma_a}(p) \equiv \exp\left(\frac{8\pi G_N \gamma_a}{L_a^2}\, i\, p_a\right),$$
$$U_{\gamma_b}(p) \equiv \exp\left(\frac{8\pi G_N \gamma_b}{L_b}\, i\, p_b\right), \tag{6}$$

where γ is a real parameter and L fixes the unit of length. The parameter γ is necessary to separate the momentum point in the phase space. We will fix the parameters γ_a and γ_b in the Section 1 using the lower eigenvalue of the area spectrum from full loop quantum gravity. Those variables can be seen as the momentum analog of the holonomy variables of loop quantum gravity.
We have also that

$$\{x_a, U_{\gamma_a}(p_a)\} = 8\pi G_N \frac{i\,\gamma_a}{L_a^2} U_{\gamma_a}(p_a),$$
$$\{x_b, U_{\gamma_b}(p_b)\} = 8\pi G_N \frac{i\,\gamma_b}{L_b} U_{\gamma_b}(p_b),$$
$$U_{\gamma_a}^{-1}\{V^m, U_{\gamma_a}\} = (4\pi R|x_b|^2)^m\, m\, |x_a|^{m-1} \mathrm{i}\gamma_a \frac{8\pi G_N}{L_a^2} \mathrm{sgn}(x_a),$$
$$U_{\gamma_b}^{-1}\{V^n, U_{\gamma_b}\} = (4\pi R|x_a|)^n\, 2n\, |x_b|^{2n-1} \mathrm{i}\gamma_b \frac{8\pi G_N}{L_b} \mathrm{sgn}(x_b). \tag{7}$$

[2] This means that in the classical action for the Kantowski-Sachs minisuperspace model

$$S = -\frac{1}{2\,G_N}\int dt\, dr\, [a\,\dot{b}^2 + 2\,\dot{a}\,\dot{b}\,b - a] = -\frac{1}{2\,G_N}\int dt\, R\, [a\,\dot{b}^2 + 2\,\dot{a}\,\dot{b}\,b - a], \tag{5}$$

we can absorb the divergent radial cell of length $\mathcal{L}_r = R$ into the variable "a" using the rescaling $a = a'/R$.

From those relations we can construct the following quantities that we use extensively

$$\frac{|x_b|^{2/3}}{|x_a|^{2/3}} = -\frac{3\,\mathrm{i}\,L_a^2}{(4\pi R)^{1/3}\,8\pi G_N \gamma_a}\,U_{\gamma_a}^{-1}\{V^{1/3}, U_{\gamma_a}\}\,\mathrm{sgn}(x_a),$$
$$\frac{|x_a|^{1/4}}{|x_b|^{1/2}} = -\frac{2\,\mathrm{i}\,L_b}{(4\pi R)^{1/4}\,8\pi G_N \gamma_b}\,U_{\gamma_b}^{-1}\{V^{1/4}, U_{\gamma_b}\}\,\mathrm{sgn}(x_b),$$
$$\sqrt{|x_a|} = -\frac{\mathrm{i}\,L_b}{(4\pi R)^{1/2}\,8\pi G_N \gamma_b}\,U_{\gamma_b}^{-1}\{V^{1/2}, U_{\gamma_b}\}\,\mathrm{sgn}(x_b),$$
$$\frac{|x_a|^{1/3}}{|x_b|^{1/3}} = -\frac{3\,\mathrm{i}\,L_b}{2\,(4\pi R)^{1/3}\,8\pi G_N \gamma_b}\,U_{\gamma_b}^{-1}\{V^{1/3}, U_{\gamma_b}\}\,\mathrm{sgn}(x_b). \tag{8}$$

We use this relation in the next section into the physical quantities. We are interested to the quantity $\frac{1}{V}$ because classically this quantity can diverge as in the case of the Schwarzschild solution and can produce a singularity. The other very important operator we will consider is $1/|x_b|^6$ that corresponds to the curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ for the Schwarzschild solution. We are also interested to the Hamiltonian constraint and to the dynamics of the minisuperspace model.

3. QUANTUM THEORY

We construct the quantum theory in analogy with the procedure used in loop quantum gravity and in particular following (Ashtekar *et al.*, 2003) but with two copy of canonical variable. In our model we have two canonical pairs (x_a, p_a) or (x_b, p_b) but we recall the polymer representation (Ashtekar *et al.*, 2003) of the Weyl-Heisenberg algebra for only one canonical pair that we denote (x, p).

The polymer representation of the Weyl-Heisenberg algebra is unitarily *inequivalent* to the Schroedi-nger representation. Now we construct the Hilbert space H_{Poly}. First of all we define a graph γ as a finite number of points $\{x_i\}$ on the real line $\mathbb{R}$. We denote by Cyl_γ the vector space of function $f(k)$ $(f : \mathbb{R} \to \mathbb{C})$ of the type

$$f(k) = \sum_j f_j e^{-ix_j k} \tag{9}$$

where $k \in \mathbb{R}$, $x_j \in \mathbb{R}$ and $f_j \in \mathbb{C}$ and j runs over a finite number of integer (labelling the points of the graph). We will call cylindrical with respect to the graph γ the function $f(k)$ in Cyl_γ. The real number k is the analog of the connections in loop quantum gravity and the plane wave e^{-ikx_j} can be thought as the holonomy of the connection k along the graph $\{x_j\}$.

Now we consider all the possible graphs (the points and their number can vary from a graph to another) and we denote Cyl the infinite dimensional vector space of functions cylindrical with respect to some graph: $\mathrm{Cyl} = \bigcup_\gamma \mathrm{Cyl}_\gamma$. A basis in Cyl is given by e^{-ikx_j} with $\langle e^{-ikx_i}|e^{-ikx_j}\rangle = \delta_{x_i,x_j}$. H_{Poly} is the Cauchy completion of Cyl or more succinctly $H_{\mathrm{Poly}} = L_2(\bar{\mathbb{R}}_{\mathrm{Bohr}}, d\mu_0)$, where $\bar{\mathbb{R}}_{\mathrm{Bohr}}$ is the Bohr-compactification of $\mathbb{R}$ and $d\mu_0$ is the Haar measure on $\bar{\mathbb{R}}_{\mathrm{Bohr}}$.

The Weyl-Heisenberg algebra is represented on H_{Poly} by the two unitary operators

$$\hat{V}(\lambda)f(k) = f(k-\lambda),$$
$$\hat{U}(\mu)f(k) = e^{i\mu k} f(k), \tag{10}$$

where $\lambda, \mu \in \mathbb{R}$. In terms of eigenkets of $\hat{V}(\lambda)$ (we associate a ket $|x_j\rangle$ with the basis elements e^{-ikx_j}) we obtain

$$\hat{V}(\lambda)|x_j\rangle = e^{i\lambda x_j}|x_j\rangle,$$
$$\hat{U}(\mu)|x_j\rangle = |x_j - \mu\rangle. \tag{11}$$

It is easy to verify that $\hat{V}(\lambda)$ is weakly continuous in λ, whence exists a self-adjoint operator $\hat{x}$ such that $\hat{x}|x_j\rangle = x_j|x_j\rangle$ (Ashtekar *et al.*, 2003; Halvorson, 2001).

The operator analogy between loop quantum gravity and polymer representation is the following: the basic operator of loop quantum gravity, holonomies and electric field fluxes, are respectively analogous to the operators $\hat{U}(\mu)$ and $\hat{x}$ with commutator $[\hat{x}, \hat{U}(\mu)] = -\mu\,\hat{U}(\mu)$. The commutator is parallel to the commutator between electric fields and holonomies. As, in the polymer representation, the unitary operator $\hat{U}(\mu)$ is well defined but the operator $\hat{p}$ doesn't exist, in loop quantum gravity the holonomies operators are unitary represented self-adjoint operators but the connection operator doesn't exist. As $\hat{x}$, the electric flux operators are unbounded self-adjoint operators with discrete eigenvalues.

After this very short review on the polymer representation of the Weyl-Heisenberg algebra we return to our system. The kantowski-Sachs minisuperspace model is characterized by two canonical pairs and the Hilbert space is $H_{\mathrm{Poly}} = L_2(\bar{\mathbb{R}}^2_{\mathrm{Bohr}}, d\mu_0)$.

A basis in the Hilbert space is the tensor product

$$|\mu_a\rangle \otimes |\mu_b\rangle \equiv |e^{-i\mu_a k_a}\rangle \otimes |e^{-i\mu_b k_b}\rangle,$$
$$\langle \mu_a|\nu_a\rangle = \delta_{\mu_a,\nu_a} \qquad \langle \mu_b|\nu_b\rangle = \delta_{\mu_b,\nu_b}. \tag{12}$$

The action of the configuration operators $\hat{V}_a(\lambda_a)$ and $\hat{V}_b(\lambda_b)$ on the bases is defined by

$$\widehat{V}_a(\lambda_a)|\mu_a\rangle = e^{i\lambda_a\hat{x}_a}|\mu_a\rangle = e^{i\lambda_a\mu_a}|\mu_a\rangle,$$
$$\widehat{V}_b(\lambda_b)|\mu_b\rangle = e^{i\lambda_b\hat{x}_b/L_b}|\mu_b\rangle = e^{i\lambda_b\mu_b}|\mu_b\rangle. \tag{13}$$

Those operators are weakly continuous in λ_a, λ_b and this imply the existence of self-adjoint operators $\hat{x}_a$ and $\hat{x}_b$, acting on the basis states according to

$$\begin{aligned} \hat{x}_a|\mu_a\rangle &= \mu_a|\mu_a\rangle, \\ \hat{x}_b|\mu_b\rangle &= L_b\mu_b|\mu_b\rangle. \end{aligned} \tag{14}$$

Now we introduce the operators corresponding to the classical momentum functions U_{γ_a} and U_{γ_b} of (6). We define the action of $\hat{U}_{\gamma_a} \equiv \hat{U}(\gamma_a)$ and $\hat{U}_{\gamma_b} \equiv \hat{U}(\gamma_b)$ on the basis states using (11) and using a quantum analog of the Poisson brackets between x_a and U_{γ_a} and x_b and U_{γ_b}

$$\begin{aligned} \hat{U}_{\gamma_a}|\mu_a\rangle &= |\mu_a - \gamma_a\rangle \qquad & \hat{U}_{\gamma_b}|\mu_b\rangle &= |\mu_b - \gamma_b\rangle, \\ [\hat{x}_a, \hat{U}_{\gamma_a}] &= -\gamma_a\hat{U}_{\gamma_a} \qquad & [\hat{x}_b, \hat{U}_{\gamma_b}] &= -\gamma_b\, L_b\, \hat{U}_{\gamma_b}. \end{aligned} \tag{15}$$

Using the standard quantization procedure $[\ ,\ ] \rightarrow i\hbar\{\ ,\ \}$, and using the the first two equations of (7) we obtain

$$L \equiv L_a = L_b = \sqrt{8\pi G_N \hbar}. \tag{16}$$

3.1. Non Singular Space-time

In this section we study the inverse volume operator and the Schwarzschild curvature invariant operator. Those quantities are classically singular in $x_b = 0$. We will use the same ideas used in loop quantum cosmology (Ashtekar *et al.*, 2003; Bojowald, 2001a,b).

We recall that given a self-adjoint operator $\hat{\mathcal{O}}$ on a Hilbert space, the function $f(\hat{\mathcal{O}})$ is well defined if and only if f is a measurable function on the spectrum of $\hat{\mathcal{O}}$. In non relativistic quantum mechanics the spectrum of $\hat{r}$ is the positive real line, equipped with the standard Lebesgue measure, therefore the operator $1/\hat{r}$ is a well defined, self-adjoint operator. On the contrary, the spectrum of the operator $\hat{x}_b$ in our minisuperspace model has a discrete topology and the operator $(\hat{x}_b)^{-1}$ is not a measurable function of $\hat{x}_b$. Since $\hat{x}_b$ admits a normalized eigenvector $|\nu = 0\rangle$, the naive expression $(\hat{x}_b)^{-1}$ fails to be densely defined on the Hilbert space. This is the case also for the operators $\widehat{\det(E)}$ and its inverse $\widehat{\frac{1}{\det(E)}}$. A similar problem arises in the full loop quantum gravity theory and can be resolved using a strategy due to Thiemann (1996, 1998a,b); Gambini *et al.* (1998). We will follow the same strategy in our minisuperspace model. First we note that we can express $\widehat{\frac{1}{\det(E)}}$ and $\widehat{\frac{1}{x_b}}$ in terms of the Poisson brackets between the volume and the fundamental variables U_{γ_a} and U_{γ_b}, then, when we quantizing the theory, we replace the Poisson brackets with $i\hbar$ times the commutator.

We use Eq. (4) and in particular the form $V = 4\pi R|x_a|\,|x_b|^2$. So, the action of the volume operator on the basis states is

$$\hat{V}|\mu, \nu\rangle = 4\pi R\,|\hat{x}_a|\,|\hat{x}_b|^2|\mu, \nu\rangle = 4\pi RL^2\,|\mu|\,|\nu|^2|\mu, \nu\rangle. \tag{17}$$

Now we show that, in the quantum theory, the operator $1/\det(E) = 1/\sqrt{h} \sim 1/|x_a|\,|x_b|^2$ does not diverge at the classical singularity point $x_b = 0$. We use the relations (8) and we promote the Poisson brackets to commutators. In this way we obtain the operator

$$\widehat{\frac{1}{\det(E)}} = \left(\widehat{\frac{|x_a|}{|x_b|^2}}\right)^3_{\gamma_b} \left(\widehat{\frac{|x_b|^2}{|x_a|^2}}\right)^3_{\gamma_a} \left(\widehat{\frac{|x_a|}{|x_b|}}\right)^2_{\gamma_b}. \tag{18}$$

The action of this operator on the bases states (for $\gamma_a = \gamma_b = 1$)[3] is

$$\begin{aligned}\widehat{\frac{1}{\det(E)}}|\mu, \nu\rangle &= \frac{2^6\,3^{15}}{L^2}\,|\mu|^5\,|\nu|^6\,[|\nu - 1|^{1/2} - |\nu|^{1/2}]^{12}\,||\mu - 1|^{1/3} - |\mu|^{1/3}|^9 \\ &\quad \times [|\nu - 1|^{2/3} - |\nu|^{2/3}]^6\,|\mu, \nu\rangle, \end{aligned} \tag{21}$$

where we defined $\hat{x}_a|\mu\rangle = \mu|\mu\rangle$ and $\hat{x}_b|\nu\rangle = L\,\nu|\nu\rangle$.

As we can see, the spectrum is upper bounded and so we have no singularity in the quantum theory in $x_b = 0$.

The other operator we want to study is $1/|x_b|^6$. This operator corresponds to the curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim \frac{1}{x_b^6}$ for the classical Schwarzschild solution. To obtain information about the singularity at the quantum level, we consider the operator $\widehat{1/|x_b|}$.

[3] To obtain the correct values for the parameters γ_a and γ_b used in the paper ($\gamma_a = \gamma_b \sim 1$) we recall from full loop quantum gravity that the area operator spectrum is

$$\hat{A}|\psi\rangle = 8\pi\beta l_P^2 \sum_p \sqrt{j_p(j_p + 1)}|\psi\rangle, \tag{19}$$

where β is the Immirzi parameter. In our symmetric model with spatial section of topology $R \times S^2$ we have considered an elementary cell $I \times S^2$ of volume $V = 4\pi l_P|a|b^2$.

Now we consider three elementary surfaces $A_{r\phi} = 2\pi l_P|a||b|$, $A_{r\theta} = 2\pi l_P|a||b|$ and $A_{\theta\phi} = 2\pi|b|^2$ respectively bounded by the interval I and the equator of the sphere S^2 ($\theta = \pi/2$), by the interval I and a circle along the longitude, and by the equator and the longitude for $0 \le \phi \le \pi$, $0 \le \theta \le \pi$. The action of the area operators on the state $|\mu = \gamma_a, \nu = \gamma_b\rangle$ is

$$\begin{aligned}&\hat{A}_{\theta\phi}|\gamma_a, \gamma_b\rangle = 2\pi l_P^2\gamma_b^2|\gamma_a, \gamma_b\rangle \\ &\hat{A}_{r\phi}|\gamma_a, \gamma_b\rangle = \hat{A}_{r\theta}|\gamma_a, \gamma_b\rangle = 2\pi l_P^2\gamma_a\gamma_b|\gamma_a, \gamma_b\rangle.\end{aligned} \tag{20}$$

Comparing the lower eigenvalue of the full loop quantum gravity area spectrum $A_0 = 4\pi\sqrt{3}\beta l_P^2$ with (20) we obtain $\gamma_a = \gamma_b = \sqrt{2\sqrt{3}\beta} \sim 0.9$. To simplify the formulas we will take $\gamma_a \sim \gamma_b \sim 1$.

Using the relations (8), we can define the operator $\widehat{1/|x_b|}$ as

$$\widehat{\frac{1}{|x_b|}} = \left(\widehat{\frac{|x_a|}{|x_b|^2}}\right)_{\gamma_b} \left(\widehat{\frac{|x_b|^2}{|x_a|^2}}\right)_{\gamma_a} \left(\widehat{\frac{|x_a|}{|x_b|}}\right)_{\gamma_b}. \tag{22}$$

The operator $\widehat{1/|x_b|}$ is diagonal on the basis states and the spectrum (for $\gamma_a = \gamma_b = 1$) is

$$\widehat{\frac{1}{|x_b|}}|\mu, \nu\rangle = \frac{2\,3^6}{L}\,|\mu|^2\,||\mu - 1|^{1/3} - |\mu|^{1/3}|^3\,|\nu|^2\,[|\nu - 1|^{1/2} - |\nu|^{1/2}]^4$$
$$\times\,||\nu - 1|^{2/3} - |\nu|^{2/3}|^3\,|\mu, \nu\rangle \tag{23}$$

This operator does not diverge in $\nu = 0$ (or $x_b = 0$), where the classical singularity is localized.

We can conclude that the quantity $1/x_b$, inspired from the Schwarzschild curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim \frac{1}{x_b^6}$, is not divergent in quantum gravity and it is possible to extend the space-time beyond the classical singularity in $r \equiv x_b = 0$.

While the boundedness of the operator $\widehat{1/x_b}$ is physically appealing, at the classical level we have the algebraically relation $x_b\left(\frac{1}{x_b}\right) = 1$ (where we define $1/x_b$ using the relations (8)) and $\hat{x}_b$ admits a normalized eigenvector with zero eigenvalue. This classical relation should be respected in an appropriate sense. We can tolerate violations of this relation on states only in the Planck regime; the equality must be satisfied on states for large eigenvalues, in the sense that it must be $\mu \gg 1$ and $\nu \gg 1$ simultaneously (i.e., with large volume). This is the case because for large eigenvalues the spectrum of $\widehat{1/x_b}$ tends to $1/(L|\nu|)$. Thus on states representing a large volume the classical algebraic relation is preserved.[4]

In this section we have defined the operators $1/\det(E)$ and $1/|x_b|$ in (18) and (22). It is also possible to define the same operators in other classically equivalent ways. This will lead to inequivalent operators in quantum theory, since the number of factor U_{γ_a} and U_{γ_b} is different.[5] In this section we have considered the simplest

[4] We can study the operator $1/\widehat{\det(E)}$ in the same way. For this operator we have the same problem in the singular point $\nu = 0$. In this point the operators $\widehat{\frac{1}{\det(E)}}$ and $\widehat{\det(E)}$ both have zero eigenvalue and we can repeat the analysis done for the operator x_b.

[5] It is know that there are ordinament problems in the definition of the inverse volume operator and in the definition of the Hamiltonian constraint in loop quantum gravity. This freedom is not a problem, it is an asset. Until now a consistent theory of quantum gravity with a well understood low energy limit doesn't exist. The important result is that the operators (inverse volume, curvature invariant and the Hamiltonian constraint) exist and are finite in quantum theory. The correct variant of the operators could be selected by some internal consistency that has not yet considered. If we can not select the correct variant using some consistency, than we have non equivalent quantum theories with the same classical limit. The physically correct one will have to be determined by experiments (Ashtekar, 2004; Rovelli, 2004; Thiemann, 2001, 2003).

case with the smallest number of U_{γ_a}, U_{γ_b} factors. In the next section we will choose the same ordering in the Hamiltonian constraint.

In this section we have studied some operator that are singular in the classical theory. In particular we obtain that quantum gravity remove the classical singular behavior. However the ultimate test as to whether or not the classical singularity persists can be obtained only considering the Hamiltonian constraint. In the next section imposing the Hamiltonian constraint we will obtain a regular difference equation which will give an evolution beyond the classical singularity and we will conclude that the evolution does not stop at the classical singular point $r = 0$.

3.2. Hamiltonian Constraint

We said that the Hamiltonian for our system depend on two canonical couples and we report now this constraint

$$H_c = \frac{G_N\, p_a^2}{2R}\,\frac{|x_a|}{x_b^2} - \frac{G_N\, p_a\, p_b}{R}\,\frac{\mathrm{sgn}(x_b)\,\mathrm{sgn}(x_a)}{|x_b|} - \frac{R}{2G_N}\,|x_a|. \tag{24}$$

Now we quantize this Hamiltonian constraint. As we know, the operators p_a and p_b don't exist in our quantum representation and so we choose the following alternative representation for the operators p_a^2 and $p_a\, p_b$.

We start from the classical expressions

$$\begin{aligned} p_a^2 &= \frac{L_a^4}{(8\pi G_N)^2}\lim_{\gamma_a\to 0}\left(\frac{2 - U_{\gamma_a} - U_{\gamma_a}^{-1}}{\gamma_a^2}\right), \\ p_a\, p_b &= \frac{L_a^2\, L_b}{2(8\pi G_N)^2}\,\lim_{\gamma_a,\gamma_b\to 0}\left[\left(\frac{U_{\gamma_a} + U_{\gamma_b} - U_{\gamma_a}\, U_{\gamma_b} - 1}{\gamma_a\,\gamma_b}\right)\right. \\ &\quad \left. + \left(\frac{U_{\gamma_a}^{-1} + U_{\gamma_b}^{-1} - U_{\gamma_a}^{-1}\, U_{\gamma_b}^{-1} - 1}{\gamma_a\,\gamma_b}\right)\right]. \end{aligned} \tag{25}$$

We have a physical interpretation setting $\gamma_a = \gamma_b = l_F/L_{\mathrm{Phys}}$, where L_{Phys} is the characteristic size of the system and l_F is a fundamental length scale. In our case $l_F = l_P$ and $\gamma_a = \gamma_b = l_P/L_{\mathrm{Phys}}$ (see also the footnote 2).

We are ready to write the Hamiltonian constraint

$$\begin{aligned} \hat{H} &= \frac{1}{32\pi^2 G_N R^2 \gamma_a^2 \gamma_b^4}\left[2 - \hat{U}_a - \hat{U}_a^{-1}\right]\left(\hat{U}_b^{-1}[\hat{V}^{1/4}, \hat{U}_b]\right)^4 + \frac{3^6}{2^{11}\pi^5 R^4 L^4 G_N \gamma_a^7 \gamma_b^5} \\ &\quad \times \left[\left(\frac{\hat{U}_a + \hat{U}_b - \hat{U}_a\,\hat{U}_b - 1}{2}\right) + \left(\frac{\hat{U}_a^{-1} + \hat{U}_b^{-1} - \hat{U}_a^{-1}\,\hat{U}_b^{-1} - 1}{2}\right)\right] \end{aligned}$$

$$\times \left(\hat{U}_b^{-1}[\hat{V}^{1/4}, \hat{U}_b]\right)^4 \left(\hat{U}_a^{-1}[\hat{V}^{1/3}, \hat{U}_a]\right)^3 \left(\hat{U}_b^{-1}[\hat{V}^{1/3}, \hat{U}_b]\right)^3 + \\ - \frac{1}{8\pi G_N L^2 \gamma_b^2} \left(\hat{U}_b^{-1}[\hat{V}^{1/2}, \hat{U}_b]\right)^2. \tag{26}$$

Now we resolve the Hamiltonian constraint and we obtain the physical states. As in non-trivially constrained systems, we expect that the physical states are not normalizable in the kinematical Hilbert space. However, as in the full loop quantum gravity theory, we again have the triplet

$$Cyl \subset H_{\text{Poly}} \subset Cyl^* \tag{27}$$

of spaces and the physical state will be in Cyl^*, which is the algebraic dual of Cyl. A generic element of this space is

$$\langle \psi | = \sum_{\mu,\nu} \psi(\mu, \nu) \langle \mu, \nu |. \tag{28}$$

The constraint equation $\hat{H}|\psi\rangle = 0$ is now interpreted as an equation in the dual space $\langle \psi | \hat{H}^\dagger$; from this equation we can derive a relation for the coefficients $\psi(\mu, \nu)$

$$\begin{aligned} &[2\alpha(\mu, \nu) - 2\beta(\mu, \nu) - \gamma(\mu, \nu)]\, \psi(\mu, \nu) - [\alpha(\mu + \gamma_a, \nu) \\ &\quad - \beta(\mu + \gamma_a, \nu)]\psi(\mu + \gamma_a, \nu) - [\alpha(\mu - \gamma_a, \nu) + \beta(\mu - \gamma_a, \nu)]\, \psi(\mu - \gamma_a, \nu) \\ &\quad + \beta(\mu, \nu + \gamma_b)\, \psi(\mu, \nu + \gamma_b) - \beta(\mu, \nu - \gamma_b)\, \psi(\mu, \nu - \gamma_b) \\ &\quad + \beta(\mu + \gamma_a, \nu + \gamma_b)\, \psi(\mu + \gamma_a, \nu + \gamma_b) \\ &\quad - \beta(\mu - \gamma_a, \nu - \gamma_b)\, \psi(\mu - \gamma_a, \nu - \gamma_b) = 0 \end{aligned} \tag{29}$$

where the functions α, β, γ are

$$\begin{aligned} \alpha(\mu, \nu) &= \frac{L^2}{8\pi^2 R G_N \gamma_b^4 \gamma_a^2} (|\mu|^{1/4}|\nu - \gamma_b|^{1/2} - |\mu|^{1/4}|\nu|^{1/2})^4, \\ \beta(\mu, \nu) &= -\frac{L^2}{2(8\pi)^2 G_N R \gamma_b^5 \gamma_a^7} (|\mu|^{1/4}|\nu - \gamma_b|^{1/2} - |\mu|^{1/4}|\nu|^{1/2})^4 \\ &\quad \times (|\mu - \gamma_a|^{1/3}|\nu|^{2/3} - |\mu|^{1/3}|\nu|^{2/3})^3 (|\mu|^{1/3}|\nu - \gamma_b|^{2/3} - |\mu|^{1/3}|\nu|^{2/3})^3 \\ \gamma(\mu, \nu) &= \frac{R}{2G_N \gamma_b^2} (|\mu|^{1/2}|\nu - \gamma_b| - |\mu|^{1/2}|\nu|)^2. \end{aligned} \tag{30}$$

The relation (29) determines the coefficients for the physical dual state and we can interpret this states as *quantum space time* near the classical point $x_b = 0$, which corresponds to the singularity of the space time in the case of classical black hole solution. From the classical Schwarzschild solution (inside the horizon) we know that the horizon is localized in "$a = 0$". Observing the difference equation (29) we obtain that the component $\psi(\mu = 0, \nu)$ is indeterminated for any "ν" (this is very similar to loop quantum cosmology situation (Ashtekar *et al.*, 2003; Bojowald, 2001a,b)). So we can identify the boundary of the Kantowski-Sachs universe in $\mu = 0$ with the horizon of a black hole. At this point we can chose as boundary condition the value of the wave function $\psi(\mu, \nu)$ very close to the event horizon.

From the difference equation (29) we obtain physical states as combinations of a countable number of components of the form $\psi(\mu + n\gamma_a, \nu + m\gamma_a)|\mu + n\gamma_a, \nu + \gamma_b\rangle$ ($\gamma_a \sim \gamma_b \sim l_P/L_{\text{Phys}} \sim 1$ at the Plank scale where $L_{\text{Phys}} \sim l_P$ (Viqar and Oliver, 2003); another way to fix the parameters γ_a and γ_b is explain in footnote two); any component corresponds to a particular value of the volume. We can interprete b as the time and the anisotropy a as the space partial observable that defines the quantum fluctuations around the Schwarzschild solution. So the function $\psi(\mu + \gamma_a, \nu + \gamma_b)$ is the wave function of the Black Hole inside the horizon at the time $\nu + \gamma_b$ and we have a natural and regular evolution beyond the point $\nu = 0$ where the classical singularity is localized. A solution of the Hamiltonian constraint corresponds to a linear combination of black hole states for particular values of the anisotropy a at the time b.

4. CONCLUSIONS

In this work we have applied the quantization procedure of (Viqar and Oliver, 2003) to the Kantowski-Sachs space time (Christodoulakis, 2002; Halliwell and Louko, 1990; Kantowski and Sachs, 1966; Luca and Torrence, 1990) with space topology $\mathbf{R} \times \mathbf{S}^2$. This space time contains the part of Schwarzschild solution on the other side of the horizon as a particular classical solution. The quantization procedure is alternative to the Schrödinger quantization and it is suggested by loop quantum gravity.

The main results are:

1. The inverse volume operator has a finite spectrum near the point $b = 0$; in particular the operator $1/b$, which is the analog of Schwarzschild curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim \frac{1}{b^6}$, does not diverge for $b = 0$ in the quantum theory and we can conclude that the classical Schwarzschild black hole singularity disappears in quantum gravity,
2. The solution of the Hamiltonian constraint gives a discrete difference equation for the coefficients of the physical states and we can have many scenarios to connect our universe to another.

An important consequence of the quantization is that, unlike the classical evolution, the quantum evolution does not stop at the classical singularity. This work is useful if we want understand what is the mechanism to resolve the problem of the "information loss" in the process of black hole formation (Ashtekar and Bojovald, 2005).

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